The Spec of a ring (See Shaf V, 1.1 - 1.2)

Let A be a ring (always commutative w/ 1).

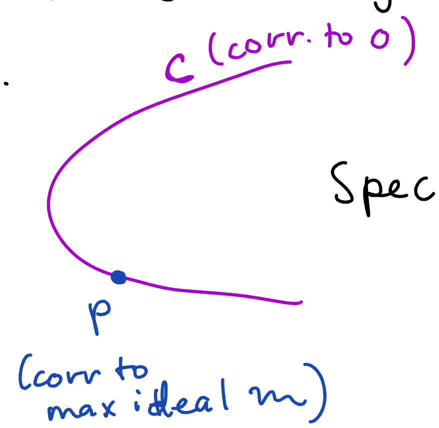
Def: The <u>spectrum</u> of A, denoted $\operatorname{Spec} A$, is the set of all prime ideals in A. The prime ideals are the <u>points</u> of $\operatorname{Spec} A$.

Ex: If $A = k[x_1, \ldots, x_n]$, $k = \bar{k}$, the <u>points</u> of $\operatorname{Spec}(A)$ correspond to the subvarieties of $\mathbb{A}^n$. (i.e. not just the points of $\mathbb{A}^n$, which only corr. to max ideals, i.e. $\operatorname{maxSpec}(A)$)

Ex: $\operatorname{Spec} \mathbb{Z} = \{(0), (2), (3), (5), (7), \ldots\}$

Ex: Let $A = (k[x])_{(x)}$, i.e. the polynomial ring localized at (x). Then the only prime ideals are (0) and (x), so $\operatorname{Spec}(A)$ is just two points

In fact, this generalizes to any local ring of a point on a curve $\mathcal{O}_p(C)$.



$$\operatorname{Spec}(\mathcal{O}_p) = \{\mathfrak{m}, (0)\}$$

Let $\varphi : A \to B$ be a ring homomorphism (which we will always require to send $1 \mapsto 1$).

Then if $P \subseteq B$ is a prime ideal, $\varphi^{-1}(P) \subseteq A$ is also a prime ideal (Exer: check this!!). So φ induces a map

$$\begin{aligned} f : \operatorname{Spec} B &\longrightarrow \operatorname{Spec} A \\ P &\longmapsto \varphi^{-1}(P) \end{aligned}$$

Ex: Consider the inclusion $\varphi : \mathbb{Z} \to \mathbb{Z}[i]$.

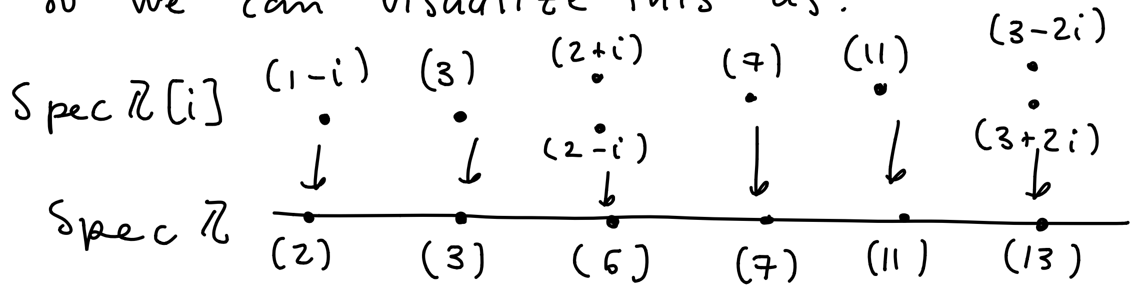
This induces $f : \operatorname{Spec} \mathbb{Z}[i] \to \operatorname{Spec} \mathbb{Z}$. (Note that $f^{-1}((0)) = (0)$)

What is $f^{-1}((p))$, for $p \in \mathbb{Z}$ prime?
It is the prime ideals $Q \subseteq \mathbb{Z}[i]$ s.t. $Q \cap \mathbb{Z} = (p)$.

$\mathbb{Z}[i]$ is a PID, so $Q = (q)$ and recall:

- If $p \equiv 3 \bmod 4$, it is prime in $\mathbb{Z}[i]$.
- If $p \equiv 1 \bmod 4$, then $p = (a+bi)(a-bi)$, some a, b,
- and if $p = 2$, then $p = i(1-i)^2$

So we can visualize this as:



Ex: Let A be a ring and $P \subseteq A$ a prime ideal. Recall that the localization

$$A_P = \left\{ \frac{a}{b} \;\middle|\; a, b \in A,\ b \notin P \right\},$$

and $\frac{a}{b} = \frac{a'}{b'}$ if $\exists\ s \notin P$ s.t. $s(ab' - a'b) = 0$.

We have a natural homomorphism

$$\begin{aligned} \varphi : A &\longrightarrow A_P \\ a &\longmapsto \frac{a}{1} \end{aligned}$$

and the prime ideals of A_P are of the form $A_P Q$ for $Q \subseteq P$. i.e.

$f : \operatorname{Spec} A_P \longrightarrow \operatorname{Spec} A$ is an inclusion and

$f^{-1}(\{Q\}) = \{A_P Q\}$.

(If you are in 523, we will be seeing this soon)

Ex: If $I \subseteq A$ is any ideal, then the primes in A/I are exactly those of the form P/I, $P \supseteq I$, so the quotient map induces the inclusion

$$\begin{aligned} \operatorname{Spec} A/I &\to \operatorname{Spec} A \\ P/I &\longmapsto P \end{aligned}$$

Local properties of Spec A

If V is an affine variety, $\Gamma(V)$ its coordinate ring, then the elements of $\Gamma(V)$ are k-valued functions on V (regular functions). By choosing coordinates, we can write $f \in \Gamma(V)$ as a polynomial, and then evaluate at each point.

However, there is a purely ring-theoretic way to do this via the residue field:

Let A be a ring and $x \in \operatorname{Spec} A$ (i.e. x is a prime ideal).

Let $\mathcal{O}_x$ be the localization of A at the ideal x. This is called the <u>local ring at x</u>, w/ max'l ideal m_x.

The <u>residue field</u> at x, denoted $k(x)$, is the quotient

$$k(x) := \mathcal{O}_x / m_x$$

So we have natural maps

$$\begin{array}{ccccc} A & \longrightarrow & \mathcal{O}_x & \longrightarrow & k(x) \\ a & \longmapsto & a/1 & \longrightarrow & \overline{a/1} \end{array}$$

and the kernel of $A \longrightarrow k(x)$ is $x \subseteq A$.

If k is alg. closed, and $A = \Gamma(V)$ for some affine variety V, and x a max'l ideal, then $k(x) = k$, by the Nullstellensatz.

Ex: If $x = (x_1, x_2 - 1, x_3 - 2) \subseteq A = k[x_1, x_2, x_3]$, then $k(x) = A/x = k$.

If we take an elt in $\Gamma(\mathbb{A}^3) = k[x_1, x_2, x_3]$, say $f = x_1^2 + x_2^2 + x_3$, then the image of f in $k(x)$ is $f(x) = 0^2 + 1^2 + 2 = 3$.

That is, the map $A \to k(x)$ is evaluation at x in this case, and in this way, f is a function on the points of $\mathbb{A}^3$, i.e. the max ideals in A.

What if we choose a non-maximal prime?

Let $q = (x_1)$. Then $k(q) = k[x_1, x_2, x_3]_q / (x_1)_q$

$\cong k[x_2, x_3]_{(0)}$, i.e. field of fractions.

So the image of f in $k(q)$ is $x_2^2 + x_3$,

so it's not taking a value in k.

More generally, for an arbitrary ring A, $f \in A$ still can be "evaluated" at each point $x \in \operatorname{Spec} A$ via

$$x \longmapsto f(x) \in k(x),$$

but at each point x, it takes a value in a different set.

Ex: $A = \mathbb{Z}$. Let $(p) \in \operatorname{Spec} \mathbb{Z}$. Then we can "evaluate" $a \in A$ at (p), $p \neq 0$ by

$$\mathbb{Z} \longrightarrow \mathbb{Z}/(p) = k((p))$$
$$a \longmapsto \bar{a} \quad \text{(residue mod } p)$$

If $(p) = (0)$, then $k((0)) = \mathbb{Z}_{(0)} = \mathbb{Q}$, so

$$a \longmapsto a/1 .$$

In classical AG, the elements of $\Gamma(V)$, where V is an affine variety, are uniquely determined by their values at the points of V. However, this is no longer true in the general case:

Ex: Let $A = k[x]/(x^2)$. Then $\operatorname{Spec} A = \{(x)\}$.

Consider the "function" $x \in A$. Then, evaluating at (x), we get

$$A \longrightarrow k[x]/(x) \cong k$$
$$x \longmapsto 0$$

so x is the zero function on $\operatorname{Spec} A$, even though $x \neq 0$.

In general, this will be the case for every nilpotent element of A (i.e. $f \in A$ s.t. $f^n = 0$ for some n).

Regular points and tangent spaces

Recall from CA:

A ring R is Noetherian if given any chain of ideals

$$I_0 \subseteq I_1 \subseteq \dots,$$

there is some n s.t. $I_n = I_{n+1} = \dots$

R is local if it has a unique max'l ideal m.

We say that a chain of prime ideals

$$p_0 \subsetneq p_1 \subsetneq \dots \subsetneq p_n$$

in R has length n, and the dimension of R is the supremum of lengths of all chains of prime ideals.

So e.g. $\dim k[x] = 1$, $\dim k = 0$. More on this later.

A Noetherian local ring (R, m) is a
ring
max'l ideal

<u>regular local ring</u> if

$$\min \# \text{ of generators of } m = \dim R.$$
$$\left(\dim_k m/m^2, \text{ where } k = R/m\right)$$

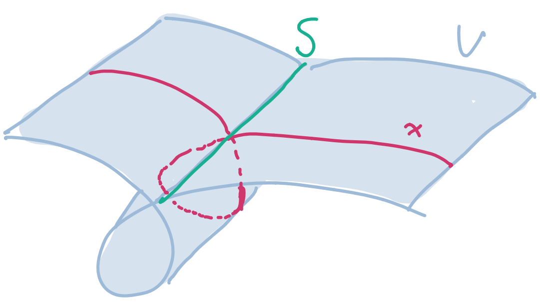
Now we can define the purely ring-theoretic formulation of regular points:

Def: A point $x \in \operatorname{Spec} A$ is regular (or nonsingular) if the local ring $\mathcal{O}_x$ is a regular local ring.

Note: For $A = \Gamma(V)$, V an affine variety, this agrees w/ our usual def. of nonsingularity at points (i.e. their corresponding max'l ideals)

For $\operatorname{Spec} A$, if $x \in \operatorname{Spec} A$ is not maximal, we will later see that regularity of x means the corresponding variety is not contained in the locus of singular points of V.

Ex:



$x \in \operatorname{Spec} A$ is a nonsingular "point" here, because the corresponding variety is not contained in the sing. locus S.

If $x \in \operatorname{Spec} A$ is regular, then the tangent space has the same dim as $\mathcal{O}_x$:

Def: Let $x \in \operatorname{Spec} A$, and $\mathcal{O}_x$ Noetherian w/ max'l ideal $\mathfrak{m}_x$. The <u>Zariski tangent space</u> to $\operatorname{Spec} A$ at x is

$$T_x = \operatorname{Hom}_{k(x)}\left(\mathfrak{m}_x/\mathfrak{m}_x^2,\ k(x)\right),$$

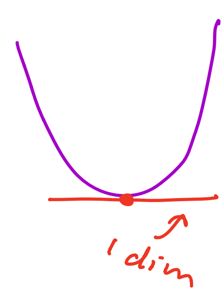
i.e. the dual vector space to $\mathfrak{m}_x/\mathfrak{m}_x^2$.

Ex: Let $A = k[x,y]/(x^2-y)$. What is the tangent space to $\operatorname{Spec} A$ at (x,y)?

$k(x) = A/(x,y) = k$, and $\mathfrak{m}_{(x,y)} = (x,y) = (x, x^2) = (x)$.

So $\mathfrak{m}_{(x,y)}/\mathfrak{m}_{(x,y)}^2 = (x)/(x^2)$, which is 1-dim.

Thus, $\dim T_{(x,y)} = 1$, as expected.

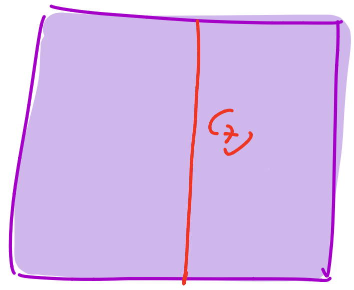


Ex: What is the dim of $T_{(x)}$ in $\operatorname{Spec} k[x,y]$?

$k((x)) = \left(k[x,y]/(x)\right)_{(x)} \cong k[y]_{(0)}$, i.e. the field of fractions.

$\mathfrak{m}_{(x)}/\mathfrak{m}_{(x)}^2 = (x)/(x^2)$, which is 1-dim over $k((x))$,

so $\dim T_{(x)} = 1$



Ex: In $\operatorname{Spec}\mathbb{Z}$, the points are of the form (0) or (p) for $p \in \mathbb{Z}$ some prime.

If $x = (0)$, Then $\mathcal{O}_x = \mathbb{Z}_{(0)} = \mathbb{Q} = k(x)$.
$\mathfrak{m}_x = (0)$, so T_x is 0-dimensional, and x is regular.

If $x = (p)$, then $\mathcal{O}_x = \mathbb{Z}_{(p)}$ and $k(x) = \mathbb{Z}/(p)$.

So $T_x = \operatorname{Hom}_{\mathbb{Z}/(p)}\left((p)/(p^2), \mathbb{Z}/(p)\right)$, which has dim 1 (check This!).

We will come back to this in a few weeks.